Calculating the number of chords and chord progressions

By Codrin Teodor Vulpoiu

vulpiusteodor@gmail.com
teodorvulpius.com
+436601188486

Abstract

This paper gives clear definitions and provides proof for calculating the number of chords and different pitch-class sets that are based on the notion of chords — chord type permutations, chord type inversions and chord types, which have the equivalent mathematical terms of linear combinations, circular permutations, circular combinations and binary necklaces with fixed content respectively. The proof relates to the applications of these combinatorial objects in music and does not employ notions from group theory. Further musically useful properties such as the degree of symmetry and the relationship between symmetry and coprimality were also revealed through this method. The paper also provides proof of the formulas for calculating the number of chord type progressions and demonstrates an efficient way of navigating through the vast number of possibilities.

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1. Chords

Let us begin by formulating the definition of a chord, according to most music dictionaries:

A chord is a simultaneous sounding of at least three notes.

Some dictionaries mention the sounding of at least two notes, which is also named an interval. We will include intervals as chords in the course of this paper.

1.1 Number of chords

According to the definition, the simultaneous combination of the notes C2, C3 and C4 qualifies as a chord, even though it only consists of octaves, which makes it very consonant. At the same time, the combination C3, C#3 and D3 is also a chord, although it is very dissonant. Taking this into account, the number of chords, as they are defined, is very large and depends on the range of the given instrument (there are more chords on a piano with 88 keys than a synth keyboard with 61 keys). But we would only like to distinguish between major and minor for example, without octaves, transposition or inversion. For this we need more precise terms that take into account these conditions. Luckily there are already mathematical terms that provide a complete description of these objects and we can draw the connection between them and the musical objects.

Let us examine the simple case of calculating the number of 3-note chords on an 88 keys piano. For this we need to understand the combinatorial notions of permutation and combination.

Definition 1.1 A permutation is an arrangement of objects in linear order.

Definition 1.2 A combination is an arrangement of objects, such that the ordering of the objects is not taken into account. It is a partition of the permutation set, a set of equivalence classes of permutations.

Lemma 1.1 Let $P(n)$ be the number of permutations for n objects. Then $P(n)=n!$.

Proof . *To arrange one object, there are n possibilities. For each of these possibilities there remain n-1 possibilities to order the second object. By induction, to order the nth object there are n-n+1 possibilities. Therefore* $P(n)=n(n-1)(n-2)\ldots(n-n+1)=n!$

Q.E.D.

Lemma 1.2 Let $C(n)$ be the number of combinations for n objects. Then $C(n)=1$.

Proof. *Following Definition 1.2 we need to eliminate the linear ordering of the objects from all possible arrangements of n by making them equivalent. But there are n! possible orderings of the objects. Therefore, we need to divide P(n) by n!, which is* $n!/n!=1$.

Q.E.D.

Corollary 1.1 Let P(n) be the set of permutations of n objects and C(n) the set of combinations. The following statement follows directly from Definition 1.2: for any element $a \in C(n)$, $C(n)=\{x \in P(n) : x \sim a\}$.

An example for P(3) is {123, 132, 213, 231, 312, 321} and its cardinality $P(3)=6$. Then C(3)={123~132~213~231~312~321} and its cardinality $C(3)=1$. We can now create a representative for this set of equivalent elements by taking, for example, the first character from the given alphabet (in our example the alphabet consists of the numbers 1, 2 and 3). Then C(3)={111}.

Lemma 1.3 Suppose there are *k* objects of one type and *n-k* objects of the second type. Let the number of combinations for *k* be the binomial coefficient , denoted by *C*(*n, k*). Then, *C*(*n, k*)=*n*!/(*k*!(*n-k*)!).

Proof. *Following the previous proof, there are n! ways to order n=k+n-k objects linearly. From those we eliminate all the orderings of the objects, thus obtaining the number of combinations of k objects, as described in Definition 1.2 and Lemma 1.2. That is k! orderings of k objects, (n-k)! orderings of n-k objects. Let* P(4, 2) *be the set of all permutations of k=2 objects consisting of the numbers 1 and 2, and n-k=2 objects consisting of the characters a and b. Then* P(4, 2)=*{12ab, 12ba, 21ab, 21ba, 2a1b, 2b1a, 2ab1, 2ba1, 1a2b, 1b2a, 1ab2, 1ba2, a12b, a21b, a1b2, a2b1, ab12, ab21, b12a, b21a, b1a2, b2a1, ba12, ba21}. Following Corollary 1.1, we can create* C(4, 2) *by making the orderings equivalent and then represent those equivalent elements by replacing the characters of each alphabet with the first characters from the given alphabets. Therefore,* C*(4,2)={12ab~12ba~21ab~21ba, 2a1b~2b1a~1a2b~1b2a, 2ab1~2ba1~1ab2~1ba2, a12b~a21b~b12a~b21a, a1b2~a2b1~b1a2~b2a1, ab12~ab21~ba12~ba21}. But {12ab~12ba~21ab~21ba}≡{11aa}, {2a1b~2b1a~1a2b~1b2a}≡{1a1a}, {2ab1~2ba1~1ab2~1ba2}≡{1aa1}, {a12b~a21b~b12a~b21a}≡{a11a}, {a1b2~a2b1~b1a2~b2a1}≡{a1a1} and {ab12~ab21~ba12~ba21}≡{aa11}. Therefore* C*(4, 2)={11aa, 1a1a, 1aa1, a11a, a1a1, aa11}. As we can see, for each permutation of k there are (n-k)! permutations of n-k and for each permutation of n-k there are k! permutations of k.*

There are therefore k!(n-k)! permutations that are equivalent in a given class. We now divide the total n! permutations by k!(n-k)! number of elements in an equivalence class to obtain the number of equivalence classes that can each be represented by one element consisting of the first characters.

Q.E.D.

In the musical context, *n* is the total number of notes that are available, *k* is the number of notes that we choose and *n-k* is the number of notes that are not chosen or played. In our piano example, n=88, k=3. We need to calculate the number of chords, so the orderings of the notes should be equivalent in the sense that choosing C2-C3-C4 is the same as choosing C3-C2-C4 because they are played simultaneously. Therefore, we need to compute the number of linear combinations, which is *C*(88, 3)=88!/(3!(88-3)!)=109736 chords with three notes.

1.2 Transpositional equivalence — chord type permutations

But we, as musicians, can recognize the octave fairly easily, so we would like to eliminate octave intervals within chords, as well as transpositions of chords (C maj is equivalent to C# maj). We thus arrive at the concept of pitch-class, which takes the whole sets of notes, such as {C0, C1, C2,…}, and treats them as one object, in our example the class of all C notes. With these pitch-classes we can form sets that are equivalent by transposition, as described earlier. We can now calculate the number of chord types (major, minor, diminished etc.) irrespective of instruments and we could call them chord type permutations. The mathematical term that describes this musical object is called circular permutation.

Definition 1.3 A circular permutation is an ordered arrangement of objects in a circular manner, as opposed to linear permutation. As a consequence of the circular arrangement, there is only one way of putting the first object, making it fixed. The ordering thus applies to the *n-1* objects.

Corollary 1.2 To arrange *n* objects in a circle, there are (n-1)! permutations.

Proof. *Following Definition 1.3 there is only one way of arranging the first object. This leaves n-1 ways of ordering the second object. By induction, there are n-n+1 ways of ordering the nth object. In total, there are 1(n-1)(n-2)…(n-n+1)=(n-1)! circular permutations for n objects.*

Q.E.D.

Lemma 1.4 To arrange *k* out of *n* objects in a circle, there are (n-1)!/(n-k)!.

Proof *The total number of ways we can arrange n objects in a circle is (n-1)!. From this we need to eliminate all the ways we can arrange the n-k objects, which is (n-k)!. By dividing (n-1)! with (n-k)!, we are left with the circular permutations of k.*

Q.E.D.

Coming back to our musical context, the total number of notes that are available to us now is 12, since we established equivalence by octave transposition, so we only have the pitches C, C#/Db, D etc. Arranging the first object is equivalent to picking any of these 12 pitches to be the fundamental of the chord, resulting in equivalent transpositions of the chords. To calculate the number of 3 note chord permutations, we compute

$P(12, 3)$=11!/9!=110 chord type permutations with 3 notes. By doing this for all numbers of notes, we obtain the following values:

Table 1. Results for calculating the number of chords

Number of notes in a chord (k)	Number of chords
2 notes	11
3 notes	110
4 notes	990
5 notes	7920
6 notes	55440
7 notes	332640
8 notes	1663200
9 notes	6652800
10 notes	19958400
11 notes	39916800
12 notes	39916800

In the case of the 3 notes chord type permutations, we have the following 6 permutations of the major triad: {tonic—third—fifth, third—fifth—tonic, fifth—tonic—third, tonic—fifth—third, fifth—third—tonic, third—tonic—fifth}. We recognize the first three permutations to be the root position, first inversion and second inversion respectively. The other three permutations have larger consecutive intervals and no conventional names. If we only want to select the compact forms of these permutations, we obtain the chord type inversions, which have the mathematical equivalent of circular combinations.

1.3 Compact form — chord type inversions

Definition 1.4 A circular combination of n objects is an unordered arrangement of objects in a circular manner. It is a partition of the circular permutation set.

Lemma 1.5 The number of circular combinations for k is $(n-1)!/((n-k)!(k-1)!)$.

Proof. *Similar to the proof in Lemma 1.3, we need to consider the number of equivalent permutations in a given equivalence class, except the first object of the k objects is now fixed, as established in Definition 1.3. This leaves (n-1)! permutations in total and (k-1)! permutations for k, as the first object is fixed. For each k-1 permutation there are (n-k)! permutations in an equivalence class. Therefore, there are (k-1)!(n-k)! permutations per equivalence class in a circular arrangement. We now divide (n-1)! total permutations by (k-1)!(n-k)! to obtain the number of equivalence classes in the circular permutations set.*

Q.E.D.

From now on, we will adopt the $C(n, k)$ notation to refer to the number of circular combinations, where an element of k is fixed. We will denote the set of circular combinations with C(n, k). An example of a set of circular combinations for k=3 and n=5 is C(5, 3)={111aa, 11a1a, 11aa1, 1a11a, 1a1a1, 1aa11}.

Bringing the three note chords example back, we can calculate the number of chord type inversions as follows: n=12, k=3, $C(12, 3)=11!/(9!2!)=55$. We now compute for the rest of *k* values and obtain the following results:

Table 2. Results for calculating the number of chord type inversions

Number of notes in a chord (*k*)	Number of chord type inversions
2 notes	11
3 notes	55
4 notes	165
5 notes	330
6 notes	462
7 notes	462
8 notes	330
9 notes	165
10 notes	55
11 notes	11
12 notes	1

This is already a big change from the previous results. The numbers are much smaller and there is some form of symmetry in the sense that $C(n, k)=C(n, n-k+1)$.

1.4 Inversional equivalence — chord types

As musicians, there is one more equivalence we would like to make, which is the equivalence of inversion (or rotation as it is called in maths). We would like to obtain the number of chord types without inversions or transpositions (ex. major, minor, diminished, augmented etc.). This leads us to the mathematical concept of necklaces, more specifically binary necklaces.

Definition 2.1 A binary necklace of length n is a string of n characters, each of 2 possible types. Rotation is ignored, in the sense that $b_1b_2...b_n$ is equivalent to $b_ib_{i+1}...b_nb_1b_2...b_{i-1}$ for any i.

Definition 2.2 A necklace with fixed content has the number of each type of character k, n-k predetermined or fixed.

Definition 2.3 A periodic or rotationally symmetric necklace of length n is a string that can be divided into repeating subsequences, thus having less than n distinct rotations.

Definition 2.4 An aperiodic necklace of length n has n distinct rotations. Each aperiodic necklace can be represented by a Lyndon word, which is the lexicographically smallest string of those that are equivalent by rotation.

Let us now examine the connection between circular combinations and necklaces. According to Definition 2.1 and 2.4, fixed density binary necklaces are strings of n characters of 2 alphabets, such that rotation is ignored and k is known. But circular combinations, after replacing the equivalence classes with representatives, also become strings of length n with 2 alphabets, where k is known. This makes the set of fixed content necklaces a partition of the circular combinations set, such that elements in the set are equivalent by rotation. An example of an equivalence class is {1100~1001}. But in the set of circular combinations, the first character in the string always belongs to k, as we have established in Lemma 1.5. This means that the maximum amount of elements in an equivalence class by rotation is k. By Definition 2.4, an aperiodic necklace has n distinct rotations. But we have already established that in an equivalence class there can be a maximum of k elements. Therefore, the necklace set is a partition of the circular combination set. An aperiodic necklace has precisely k elements in an equivalence class and, consequently, a rotationally symmetric or periodic necklace has less than k elements. According to Definition 2.4, it has been established that the lexicographically smallest element will be chosen as a representative for a given equivalence class. For example, {1100~1001}≡{1100}.

We will adopt the conventional N(n, k) notation to refer to the set of necklaces. The notation L(n, k) is referring to aperiodic necklaces ("L" stands for Lyndon word) and we will write R(n, k) to denote the set of rotationally symmetric necklaces. Their notation with italic letters will denote their cardinality. To distinguish between equivalence classes with k and less than k elements, we will denote the set of circular combinations that belong to aperiodic necklaces with $\mathrm{C}^{\mathrm{L}}(n, k)$, and the circular combinations that belong to rotationally symmetric necklaces with $\mathrm{C}^{\mathrm{R}}(n, k)$.

Corollary 2.1 Given the definitions and relations discussed so far, the following statements can be made as a consequence:

$\mathrm{L}(n, k)=\mathrm{C}^{\mathrm{L}}(n, k)/\sim$

$\mathrm{R}(n, k)=\mathrm{C}^{\mathrm{R}}(n, k)/\sim$

$\mathrm{N}(n, k)=\mathrm{C}(n, k)/\sim$

$L(n, k)+R(n, k)=N(n, k)$

$C^L(n, k)+C^R(n, k)=C(n, k)$

$L(n, k)=C^L(n, k)/k$

Lemma 2.1 $N(n, k)=L(n, k)=C(n, k)/k \Leftrightarrow \gcd(n, k)=1$.

Proof. *According to Definition 2.3, periodic necklaces can be divided into repeating subsequences. A necklace can be divided into repeating subsequences if, and only if, it can be divided into equal parts. A string of characters can be divided into equal parts if, and only if, the number of each type of character can be equally divided by some number larger than 1. Suppose you divide k by 2 and n-k by 3 to form equal subsequences. Then there will be 3 strings, where at least one contains no k objects, which is a contradiction with the initial statement. By induction, you cannot divide each k, n-k by unequal amounts to form identical subsequences, therefore* $\mathrm{N}(n, k)=\mathrm{L}(n, k)$ and $N(n, k)=L(n, k)$. *But in Corollary 2.1 we have obtained* $\mathrm{N}(n, k)=\mathrm{C}(n, k)/\sim$ *, and if the representatives of the equivalence classes cannot be divided into repeating subsequences, then all elements of* $\mathrm{C}(n, k)$ *cannot be divided into repeating subsequences. Therefore,* $\mathrm{C}(n, k)=\mathrm{C}^{\mathrm{L}}(n, k)$ and $C(n, k)=C^L(n, k) \Leftrightarrow gcd(n, k)=1$. *But also in Corollary 2.1, we have obtained that* $L(n, k)= C^L(n, k)/k$ *, therefore,*
$N(n, k)=C(n, k)/k \Leftrightarrow gcd(n, k)=1$.

Q.E.D.

Lemma 2.2 For k or n-k is a prime number and $\gcd(n, k)\neq1$, $C^R(n, k)=R(n, k)=1$.

Proof. *Suppose k is a prime number. Then, following the proof provided by Euclid in Book VII, Proposition 29,* $gcd(n, k)\neq1 \Leftrightarrow gcd(n, k)=k$. *But a prime number can only be divided by 1 and itself and, following Lemma 2.1, a necklace is rotationally symmetric if, and* *only if, n, k can be divided by a number larger than 1. Therefore, in our case, n, k can only be divided once by k. But each of those subsequences will contain exactly 1 of k objects, since k/k=1 and the object is fixed, by Lemma 1.5.*

Q.E.D.

An example for a rotationally symmetric necklace, where k is prime, is $R(9, 3)=C^R(9, 3)=\{1aa1aa1aa\}$. Any rotation of this string, such that there is a "1" at the beginning of the string, produces its identity. The repeating substring is {1aa}.

Corollary 2.2 As a direct result of Lemma 2.2, the following statement can be made: $L(n, k)=(C(n, k)-1)/k \Leftrightarrow \gcd(n, k)\neq 1$ and k is prime.

An example for a set where all necklaces are aperiodic is N(5, 2)={11aaa, 1a1aa}.

Lemma 2.3 Let $S^R(n, k)$ be the set of repeating substrings into which periodic necklaces can be divided. Then $S^R(n, k)=\bigcup_{d|\gcd(n, k),\, d>1} L(n/d, k/d)$.

Proof. *Let there be a periodic necklace* R(6,4)={*aabaab*}. *This particular example can be divided into two identical subsequences, namely {aab}. Suppose you can divide the same periodic necklace into the following repeating subsequence {aba}. Both {aab} and {aba} are equivalent by rotation and any periodic necklaces of the same length that they can produce will also be equivalent by rotation, and will belong, as a consequence, to the same equivalence class of the circular combination set. For any periodic necklace of length n, a division of it by some number x will produce identical strings of length n/x. Any rotation of that string, when extended x times to produce the periodic necklace of length n, will produce* *a rotation of the original necklace. Therefore, the subsequences need to be necklaces and not circular combinations. We have established that, in order for a necklace to be periodic, n, k have to be divisible by some common factor $d>1$. Let $y(d)$ be the number of common factors of n, k. If $y(d)=1$, the periodic necklace can be divided one time into $d=gcd(n, k)$ subsequences. If $y(d)=2$, the periodic necklaces can be divided two times — once into $d_1=gcd(n, k)$ subsequences and once into d_2 subsequences, where $d_2|gcd(n, k)$. And so for any $y(d)$, the periodic necklace can be divided y times. If the subsequence is a periodic necklace itself, then the subsequence can be further divided by some number x. For example, the subsequence {1a1a} can be further divided into x=2 repeating subsequences {1a}. But if $x|d$ and $d|gcd(n, k)$, then $x|gcd(n, k)$, which is already taken into account by $d|gcd(n, k)$. Therefore, the repeating subsequences need to be aperiodic necklaces. In order to prove uniqueness, we proceed by solving the following two cases:*

Case 1 — two different subsequences of the same length cannot produce the same periodic necklace. We have already proved that the two subsequences cannot be rotations of each other. If the number of characters of each alphabet is different for the two subsequences, multiplying them by the same number would produce different numbers of characters of each alphabet and, consequently, different periodic necklaces. For example, extending the two subsequences {1a1aa} and {1a11a} by the same amount d=2 would produce the two different periodic necklaces {1a1aa1a1aa} and {1a11a1a11a}. Suppose that two different subsequences have the same number of each character type and they are not rotations of each other. Then the periodic necklaces that they produce are not rotations of each other and, consequently, different.

Case 2 — two aperiodic necklaces of different lengths cannot produce the same necklace. In order for the two subsequences to produce the same periodic necklace, one of them needs to be extended x amount of times and the other needs to be extended y amount of *times. So, gcd(xn, xk) of the first necklace is equal to xgcd(n, k) and gcd(yn, yk) of the second must equal ygcd(n, k). But gcd(n, k)=1 for both subsequences, since they are aperiodic, so in order to produce the same periodic necklace, gcd(xn, xk)=gcd(yn, yk), and therefore x=y, which is absurd.*

Q.E.D.

Corollary 2.3 As a direct consequence of Lemma 2.3 and of the previous proofs, the following formulas can be made:

$R(n, k)=\sum_{d|\gcd(n, k),\, d>1} L(n/d, k/d)$

$C^R(n, k)=\sum_{d|\gcd(n, k),\, d>1} C^L(n/d, k/d)$

At this point we have a complete set of formulas for counting necklaces with fixed content:

$C(n, k)=(n-1)!/((k-1)!(n-k)!)$

$C^L(n, k)=C(n, k)-C^R(n, k)$

$C^R(n, k)=\sum_{d|\gcd(n, k),\, d>1} C^L(n/d, k/d)$

$L(n, k)=(C(n, k)-C^R(n, k))/k$

$R(n, k)=\sum_{d|\gcd(n, k),\, d>1} L(n/d, k/d)$

$N(n, k)=L(n, k)+R(n, k)$

This method, although complete, is recursive. We will now proceed with proving the simplification of these formulas.

Proposition 1.1 For $\gcd(n, k)=p^a$, where p is a prime number and a is any positive integer, $C^R(n, k)=C(n/p, k/p)$.

Proof. *According to the fundamental theorem of arithmetic, any number that is not prime is a unique product of prime numbers. This also applies to the gcd(n, k) and this recurring division will carry out until n and k are coprime. Using the recursive formulas obtained in Corollary 2.3,* $C^R(n, k)=C(n/p, k/p)-C(n/p^2, k/p^2)+C(n/p^2, k/p^2)-\ldots-C(n/p^{a-1}, k/p^{a-1})+C(n/p^{a-1}, k/p^{a-1})-C(n/p^a, k/p^a)+C(n/p^a, k/p^a)$. *This summation stops with* $C(n/p^a, k/p^a)$ *because* n/p^a, k/p^a *are guaranteed to be coprime by the fact that* $gcd(n, k)=p^a$. *But* $-C(n/p^2, k/p^2)+C(n/p^2, k/p^2)=0$, $-C(n/p^3, k/p^3)+C(n/p^3, k/p^3)=0$, *all the way to* $-C(n/p^{a-1}, k/p^{a-1})+C(n/p^{a-1}, k/p^{a-1})=0$ *and* $-C(n/p^a, k/p^a)+C(n/p^a, k/p^a)=0$. *Therefore, the only term that remains from this expression is* $C(n/p, k/p)$.

Q.E.D.

Proposition 1.2 For $\gcd(n, k)=p_1^a p_2^b$, where p_1 and p_2 are the unique prime factors raised to the power of a and b respectively,

$C^R(n, k)=C(n/p_1, k/p_1)+C(n/p_2, k/p_2)-C(n/(p_1p_2), k/(p_1p_2))$.

Proof. *Let* $d(n, k)$ *be the set of* $d|gcd(n, k)$, *such that* $d>1$. *For the case where* $gcd(n, k)=p_1^a p_2^b$, $d(n, k)=\{p_1^1, p_1^2, \ldots, p_1^a, p_2^1, p_2^2, \ldots, p_2^b, p_1^1p_2^1, p_1^2p_2^2, \ldots, p_1^ap_2^b, p_1^1p_2^2, \ldots, p_1^1p_2^b, p_1^2p_2^3, \ldots, p_1^2p_2^b, \ldots, p_1^{a-1}p_2^b, p_1^2p_2^1, \ldots, p_1^ap_2^1, p_1^3p_2^2, \ldots, p_1^ap_2^2, \ldots, p_1^ap_2^{b-1}\}$. *We can construct a matrix with all elements of* $d(n, k)$.

	p_1	p_1^2	…	p_1^{a-2}	p_1^{a-1}	p_1^a
p_2	p_1p_2	$p_1^2p_2$	…	$p_1^{a-2}p_2$	$p_1^{a-1}p_2$	$p_1^ap_2$
p_2^2	$p_1p_2^2$	$p_1^2p_2^2$	…	$p_1^{a-2}p_2^2$	$p_1^{a-1}p_2^2$	$p_1^ap_2^2$

⋮	⋮	⋮	...	⋮	⋮	⋮
p_2^{b-2}	$p_1p_2^{b-2}$	$p_1^2p_2^{b-2}$	...	$p_1^{a-2}p_2^{b-2}$	$p_1^{a-1}p_2^{b-2}$	$p_1^ap_2^{b-2}$
p_2^{b-1}	$p_1p_2^{b-1}$	$p_1^2p_2^{b-1}$	...	$p_1^{a-2}p_2^{b-1}$	$p_1^{a-1}p_2^{b-1}$	$p_1^ap_2^{b-1}$
p_2^b	$p_1p_2^b$	$p_1^2p_2^b$	...	$p_1^{a-2}p_2^b$	$p_1^{a-1}p_2^b$	$p_1^ap_2^b$

This recurrence relation can continue until gcd(n, k)/d=1 and no further division can occur, because gcd(n/d, k/d)=1. But {x : x=gcd(n, k)/d} ≡ {δ : δ|gcd(n, k), δ<gcd(n, k)}, so by dividing gcd(n, k)/d, we obtain the following equivalent matrix, which should make visualisation of the proof easier:

	$p_1^{a-1}p_2^b$	$p_1^{a-2}p_2^b$	...	$p_1^2p_2^b$	$p_1p_2^b$	p_2^b
$p_1^ap_2^{b-1}$	$p_1^{a-1}p_2^{b-1}$	$p_1^{a-2}p_2^{b-1}$	...	$p_1^2p_2^{b-1}$	$p_1p_2^{b-1}$	p_2^{b-1}
$p_1^ap_2^{b-2}$	$p_1^{a-1}p_2^{b-2}$	$p_1^{a-2}p_2^{b-2}$	...	$p_1^2p_2^{b-2}$	$p_1p_2^{b-2}$	p_2^{b-2}
⋮	⋮	⋮	...	⋮	⋮	⋮
$p_1^ap_2^2$	$p_1^{a-1}p_2^2$	$p_1^{a-2}p_2^2$	...	$p_1^2p_2^2$	$p_1p_2^2$	p_2^2
$p_1^ap_2$	$p_1^{a-1}p_2$	$p_1^{a-2}p_2$	...	$p_1^2p_2$	p_1p_2	p_2
p_1^a	p_1^{a-1}	p_1^{a-2}	...	p_1^2	p_1	1

Then using the same recursive formula from Corollary 2.3,

$C^R(n, k)=\sum_{d\in \mathrm{d}(n,k)} C(n/d, k/d)-C^R(n/d, k/d)$. We now want to find how many C(n/d, k/d) terms there are in this recurrence relation without computing the circular combination function. This recurrence relation begins by summing all the gcd(n, k)/d terms, which are equivalent to

the δ elements that can be seen in the second matrix. Each of those elements, except for 1, are further divisible by some factor d>1. So from each of those elements we need to subtract $C^R(gcd(n, k)/d)$. *But* $C^R(gcd(n, k)/d)=C(gcd(n, k)/(dg))$, *where* $g \in d(n, k)$. *In the second matrix this can be visualised such that each term is divisible by the terms right and below it and from each term, all the terms right and below it need to be subtracted, and from those again the same, until reaching 1 and the recurrence relation ends.*

For $gcd(n, k)=p_1^1p_2^1$, $C^R(n, k)=C(n/p_1, k/p_1)-C(n/(p_1p_2), k/(p_1p_2))+C(n/p_2, k/p_2)-C(n/(p_1p_2), k/(p_1p_2))+C(n/(p_1p_2), k/(p_1p_2))=C(n/p_1, k/p_1)+C(n/p_2, k/p_2)-C(n/(p_1p_2), k/(p_1p_2))$.

For $gcd(n, k)=p_1p_2^2$, $C^R(n, k)=C(n/p_1, k/p_1)-C^R(n/p_1, k/p_1)+C(n/p_2, k/p_2)-C^R(n/p_2, k/p_2)+C(n/p_2^2, k/p_2^2)-C^R(n/p_2^2, k/p_2^2)+C(n/(p_1p_2), k/(p_1p_2))-C^R(n/(p_1p_2), k/(p_1p_2))+C(n/(p_1p_2^2), k/(p_1p_2^2))$.

But in Proposition 1.1 we have demonstrated that $C^R(n, k)=C(n/p, k/p) \Leftrightarrow gcd(n, k)=p^a$. *So,* $C^R(n/p_2^2, k/p_2^2)=C^R(n/(p_1p_2), k/(p_1p_2))=C(n/(p_1p_2^2), k/(p_1p_2^2))$ *and* $C^R(n/p_1, k/p_1)=C(n/(p_1p_2), k/(p_1p_2))$. *Also,* $C^R(n/(p_1p_2^2), k/(p_1p_2^2))=0$. *We have also previously demonstrated that* $C^R(n/p_2, k/p_2)=C(n/p_2^2, k/p_2^2)+C(n/(p_1p_2), k/(p_1p_2))-C(n/(p_1p_2^2), k/(p_1p_2^2))$. *So,* $C^R(n, k)=C(n/p_1, k/p_1)-C(n/(p_1p_2), k/(p_1p_2))+C(n/p_2, k/p_2)-C(n/p_2^2, k/p_2^2)-C(n/(p_1p_2), k/(p_1p_2))+C(n/(p_1p_2^2), k/(p_1p_2^2))+C(n/p_2^2, k/p_2^2)-C(n/(p_1p_2^2), k/(p_1p_2^2))+C(n/(p_1p_2), k/(p_1p_2))-C(n/(p_1p_2^2), k/(p_1p_2^2))+C(n/(p_1p_2^2), k/(p_1p_2^2))=C(n/p_1, k/p_1)+C(n/p_2, k/p_2)-C(n/(p_1p_2), k/(p_1p_2))$. *The same method applies to* $gcd(n, k)=p_1^2p_2$.

For $gcd(n, k)=p_1^2p_2^2$, $C^R(n, k)=C(n/p_1, k/p_1)-C^R(n/p_1, k/p_1)+C(n/p_2, k/p_2)-C^R(n/p_2, k/p_2)+C(n/p_1^2, k/p_1^2)-C^R(n/p_1^2, k/p_1^2)+C(n/p_2^2, k/p_2^2)-C^R(n/p_2^2, k/p_2^2)+C(n/(p_1^2p_2), k/(p_1^2p_2))-C^R(n/(p_1^2p_2), k/(p_1^2p_2))+C(n/(p_1p_2^2), k/(p_1p_2^2))-C^R(n/(p_1p_2^2), k/(p_1p_2^2))+C(n/(p_1p_2), k/(p_1p_2))-C^R(n/(p_1p_2), k/(p_1p_2))+C(n/(p_1^2p_2^2), k/(p_1^2p_2^2))$.

We have already proved the formula for each C^R *term, therefore* $C^R(n, k)=C(n/p_1, k/p_1)-C(n/p_1^2, k/p_1^2)+C(n/(p_1^2p_2), k/(p_1^2p_2))-C(n/(p_1p_2), k/(p_1p_2))+$

$C(n/(p_1^2p_2), k/(p_1^2p_2))+C(n/(p_1p_2^2), k/(p_1p_2^2))-C(n/(p_1^2p_2^2), k/(p_1^2p_2^2))-C(n/(p_1p_2^2), k/(p_1p_2^2))+$

$C(n/(p_1^2p_2^2), k/(p_1^2p_2^2))-C(n/(p_1^2p_2), k/(p_1^2p_2))+C(n/(p_1^2p_2^2), k/(p_1^2p_2^2))-$

$C(n/(p_1^2p_2^2), k/(p_1^2p_2^2))+C(n/p_2, k/p_2)-C(n/p_2^2, k/p_2^2)+C(n/(p_1p_2^2), k/(p_1p_2^2))-$

$C(n/(p_1p_2), k/(p_1p_2))+C(n/(p_1^2p_2), k/(p_1^2p_2))+C(n/(p_1p_2^2), k/(p_1p_2^2))-C(n/(p_1^2p_2^2), k/(p_1^2p_2^2))-$

$C(n/(p_1p_2^2), k/(p_1p_2^2))+C(n/(p_1^2p_2^2), k/(p_1^2p_2^2))-C(n/(p_1^2p_2), k/(p_1^2p_2))+C(n/(p_1^2p_2^2), k/(p_1^2p_2^2))-$

$C(n/(p_1^2p_2^2), k/(p_1^2p_2^2))+C(n/p_1^2, k/p_1^2)-C(n/(p_1^2p_2), k/(p_1^2p_2))+C(n/p_2^2, k/p_2^2)-$

$C(n/(p_1p_2^2), k/(p_1p_2^2))+C(n/(p_1p_2^2), k/(p_1p_2^2))-C(n/(p_1^2p_2^2), k/(p_1^2p_2^2))+C(n/(p_1^2p_2), k/(p_1^2p_2))-$

$C(n/(p_1^2p_2^2), k/(p_1^2p_2^2))+C(n/(p_1p_2), k/(p_1p_2))-C(n/(p_1^2p_2), k/(p_1^2p_2))-C(n/(p_1p_2^2), k/(p_1p_2^2))+$

$C(n/(p_1^2p_2^2), k/(p_1^2p_2^2))+C(n/(p_1^2p_2^2), k/(p_1^2p_2^2))=C(n/p_1, k/p_1)+C(n/p_2, k/p_2)-$

$C(n/(p_1p_2), k/(p_1p_2))$.

By induction, for $gcd(n, k)=p_1^ap_2^b$, *where* $a,b>1$,

$C^R(n/(p_1^ap_2^b), k/(p_1^ap_2^b))=0$, $C^R(n/(p_1^ap_2^{b-1}), k/(p_1^ap_2^{b-1}))=C(n/(p_1^ap_2^b), k/(p_1^ap_2^b))$,

$C^R(n/(p_1^{a-1}p_2^b), k/(p_1^{a-1}p_2^b))=C(n/(p_1^ap_2^b), k/(p_1^ap_2^b))$, $C^R(n/(p_1^{a-1}p_2^{b-1}), k/(p_1^{a-1}p_2^{b-1}))=$

$C(n/(p_1^ap_2^{b-1}), k/(p_1^ap_2^{b-1}))+C(n/(p_1^{a-1}p_2^b), k/(p_1^{a-1}p_2^b))-C(n/(p_1^ap_2^b), k/(p_1^ap_2^b))$,

$C^R(n/p_1^a, k/p_1^a)=C(n/(p_1^ap_2), k/(p_1^ap_2))$, $C^R(n/p_2^b, k/p_2^b)=C(n/(p_1p_2^b), k/(p_1p_2^b))$,

$C^R(n/p_1^{a-1}, k/p_1^{a-1})=C(n/(p_1^{a-1}p_2), k/(p_1^{a-1}p_2))+C(n/p_1^a, k/p_1^a)-C(n/(p_1^ap_2), k/(p_1^ap_2))$,

$C^R(n/p_2^{b-1}, k/p_2^{b-1})=C(n/(p_1p_2^{b-1}), k/(p_1p_2^{b-1}))+C(n/p_2^b, k/p_2^b)-C(n/(p_1p_2^b), k/(p_1p_2^b))$.

$C^R(n, k)=C(n/(p_1^ap_2^b), k/(p_1^ap_2^b))+(\sum_a\sum_b C(n/(p_1^{a-1}p_2^{b-1}), k/(p_1^{a-1}p_2^{b-1}))-$

$C(n/(p_1^ap_2^{b-1}), k/(p_1^ap_2^{b-1}))-C(n/(p_1^{a-1}p_2^b), k/(p_1^{a-1}p_2^b))+C(n/(p_1^ap_2^b), k/(p_1^ap_2^b)))+$

$(\sum_b C(n/(p_1^ap_2^{b-1}), k/(p_1^ap_2^{b-1}))-C(n/(p_1^ap_2^b), k/(p_1^ap_2^b)))+(\sum_a C(n/(p_1^{a-1}p_2^b), k/(p_1^{a-1}p_2^b))-$

$C(n/(p_1^ap_2^b), k/(p_1^ap_2^b)))+(\sum_a C(n/p_1^{a-1}, k/p_1^{a-1})-C(n/(p_1^{a-1}p_2), k/(p_1^{a-1}p_2))-C(n/p_1^a, k/p_1^a)+$

$C(n/(p_1^ap_2), k/(p_1^ap_2)))+(\sum_b C(n/p_2^{b-1}, k/p_2^{b-1})-C(n/(p_1p_2^{b-1}), k/(p_1p_2^{b-1}))-C(n/p_2^b, k/p_2^b)+$

$C(n/(p_1p_2^b), k/(p_1p_2^b)))+C(n/p_1^a, k/p_1^a)-C(n/(p_1^ap_2), k/(p_1^ap_2))+C(n/p_2^b, k/p_2^b)-$

$C(n/(p_1p_2^b), k/(p_1p_2^b))=C(n/p_1, k/p_1)+C(n/p_2, k/p_2)-C(n/(p_1p_2), k/(p_1p_2))$.

Q.E.D.

Proposition 1.3 For any gcd(n, k)=$p_1^a p_2^b p_3^c$, where p_1, p_2 and p_3 are unique prime factors raised to the power of a, b and c respectively,

$C^R(n, k)=C(n/p_1, k/p_1)+C(n/p_2, k/p_2)+C(n/p_3, k/p_3)+C(n/(p_1p_2p_3), k/(p_1p_2p_3))-$
$C(n/(p_1p_2), k/(p_1p_2))-C(n/(p_1p_3), k/(p_1p_3))-C(n/(p_2p_3), k/(p_2p_3))$.

Proof. *Similar to the proof in Proposition 1.2, we need to construct a matrix with gcd(n, k)/d, where d>1, with the alternative matrix containing the terms gcd(n, k)/δ in descending order, where δ|gcd(n, k) and δ<gcd(n, k). This matrix is now three-dimensional, like a rectangular prism or cube, such that it is extended from the two-dimensional matrix in Proposition 1.2 with the z-axis containing the* p_3 *terms starting from the top left corner, pointing towards the viewer. We have so far provided proof for calculating the terms on the front, right and bottom faces of the three dimensional matrix. After performing the algebraic expression which has been omitted here to save space (it can be requested from the author), the following results are obtained:*

For $gcd(n, k)=p_1^1p_2^1p_3^1$, $C^R(n, k)=C(n/p_1, k/p_1)+C(n/p_2, k/p_2)+C(n/p_3, k/p_3)+C(n/(p_1p_2p_3), k/(p_1p_2p_3))-C(n/(p_1p_2), k/(p_1p_2))-C(n/(p_1p_3), k/(p_1p_3))-C(n/(p_2p_3), k/(p_2p_3))$.

For $gcd(n, k)=p_1^2p_2^1p_3^1$, *So,* $C^R(n, k)=C(n/p_1, k/p_1)+C(n/p_2, k/p_2)+C(n/p_3, k/p_3)+C(n/(p_1^2p_2p_3), k/(p_1^2p_2p_3))-C(n/(p_1p_2), k/(p_1p_2))-C(n/(p_1p_3), k/(p_1p_3))-C(n/(p_2p_3), k/(p_2p_3))$. *The same result applies to* $gcd(n, k)=p_1^1p_2^2p_3^1$ *and* $gcd(n, k)=p_1^1p_2^1p_3^2$.

For $gcd(n, k)=p_1^2p_2^2p_3^1$, $C^R(n, k)=C(n/p_1, k/p_1)+C(n/p_2, k/p_2)+C(n/p_3, k/p_3)+C(n/(p_1p_2p_3), k/(p_1p_2p_3))-C(n/(p_1p_2), k/(p_1p_2))-C(n/(p_1p_3), k/(p_1p_3))-C(n/(p_2p_3), k/(p_2p_3))$. *The same applies to* $gcd(n, k)=p_1^1p_2^2p_3^2$ *and* $gcd(n, k)=p_1^2p_2^1p_3^2$.

By induction, for $gcd(n, k)=p_1^ap_2^bp_3^c$, *where* $a,b,c > 1$, $C^R(n, k)=C(n/p_1, k/p_1)+$

$C(n/p_2, k/p_2)+C(n/p_3, k/p_3)+C(n/(p_1p_2p_3), k/(p_1p_2p_3))-C(n/(p_1p_2), k/(p_1p_2))-$

$C(n/(p_1p_3), k/(p_1p_3))-C(n/(p_2p_3), k/(p_2p_3)).$

Q.E.D.

Corollary 3.1 We now have a complete simplified formula for calculating aperiodic necklaces with gcd(n, k) equal up to three prime factors raised to any power:

$L(n, k)=(C(n, k)-C^R(n, k))/k$, where $C^R(n, k)=C(n/p, k/p) \Leftrightarrow$ gcd$(n, k)=p^a$,

$C^R(n, k)=C(n/p_1, k/p_1)+C(n/p_2, k/p_2)-C(n/(p_1p_2), k/(p_1p_2)) \Leftrightarrow$ gcd$(n, k)=p_1^a p_2^b$,

$C^R(n, k)=C(n/p_1, k/p_1)+C(n/p_2, k/p_2)+C(n/p_3, k/p_3)+C(n/(p_1p_2p_3), k/(p_1p_2p_3))-$

$C(n/(p_1p_2), k/(p_1p_2))-C(n/(p_1p_3), k/(p_1p_3))-C(n/(p_2p_3), k/(p_2p_3)) \Leftrightarrow$ gcd$(n, k)=p_1^a p_2^b p_3^c$.

We are essentially predicting what the Möbius function would produce and using smaller terms by using circular combinations instead of linear combinations. Let us show the following example for $L(24, 12)$. Using the old formula, $L(24, 12)=(\mu(1)C(24, 12)+\mu(2)C(12, 6)+\mu(3)C(8, 4)+\mu(4)C(6, 3)+\mu(6)C(4, 2)+\mu(12)C(2, 1))/24$, where $\mu(d)$ is the Möbius function applied to $d|$gcd(24, 12) and $C(n, k)=n!/(k!(n-k)!)$. Computing the Möbius function, $\mu(1)=1$, $\mu(2)=-1$, $\mu(3)=-1$, $\mu(4)=0$, $\mu(6)=1$ and $\mu(12)=0$. Therefore,

$L(24, 12)=(24!/(12!12!)-12!/(6!6!)-8!/(4!4!)+4!/(2!2!))/24=112632$.

Let us now use our formula. For gcd(24, 12)=12=$2^2 3$, $C^R(24, 12)=C(12, 6)+C(8, 4)-C(4, 2)$, where $C(n, k)=(n-1)!/((k-1)!(n-k)!)$. Therefore,

$L(24, 12)=(23!/(11!12!)-11!/(5!6!)-7!/(3!4!)+3!/(1!2!))/12=112632$.

As we can notice, $(n!/(k!(n-k)!))/n=((n-1)!/((k-1)!(n-k)!))/k$ and we are taking the same divisors into account, therefore we can replace our formula with the Möbius function, which has already been proven by mathematicians.

Proposition 2.1 After applying the formulas from Proposition 1 to the formula in Corollary 2.3, we can arrive at the following simplified method for calculating rotationally symmetric necklaces:

$R(n, k)=1/k\sum_{d|\gcd(n, k), d>1} X(d)C(n/d, k/d)$, where

$X(d)=d \Leftrightarrow d=p^1$

$X(d)=d-p^{a-1} \Leftrightarrow d=p^a$ and $a>1$

$X(d)=d-p_1-p_2 \Leftrightarrow d=p_1^1p_2^1$

$X(d)=d- p_1^{a-1}p_2^{b-1}(p_1+p_2-1) \Leftrightarrow d=p_1^ap_2^b$ and a or $b>1$

$X(d)=d-p_1p_3-p_2p_3-p_1p_2+p_1+p_2+p_3 \Leftrightarrow d=p_1^1p_2^1p_3^1$

$X(d)=d-p_1^{a-1}p_2^{b-1}p_3^{c-1}(1+p_1p_3+p_2p_3+p_1p_2-p_1-p_2-p_3) \Leftrightarrow d=p_1^ap_2^bp_3^c$ and either a,b or $c>1$.

Proof. *Applying the formula we know thus far,*

$R(n, k)=\sum_{d|\gcd(n, k), d>1} (C(n/d, k/d)-C^R(n/d, k/d))/(n/d)$.

If we amplify each of those fractions by d, they will have k as a common denominator. This leads to $R(n, k)=1/k\sum_{d|\gcd(n, k), d>1} d(C(n/d, k/d)-C^R(n/d, k/d))$.

If we go back to the matrices we have built in Proposition 1, we need to consider which type of terms have less divisors than the preceding terms. These are to be found at the extremes of the matrix, where there are no more terms right, below or in front of them, by which they can be divided. When $\gcd(n, k)=p_1^a$, *the only special term in the line is* $C^R(n/p_1^a, k/p_1^a)=0$. *When* $\gcd(n, k)=p_1^ap_2^b$, *the special terms are* $C^R(n/(p_1^ap_2^b), k/(p_1^ap_2^b))=0$, $C^R(n/p_1^{a-1}, k/p_1^{a-1})=C(n/p_1^{a-1}, k/p_1^{a-1})-C(n/p_1^a, k/p_1^a)$ *and* $C^R(n/p_1^{b-1}, k/p_1^{b-1})=C(n/p_1^{b-1}, k/p_1^{b-1})-C(n/p_1^b, k/p_1^b)$. *We should therefore consider these special terms separately when calculating* $R(n, k)$.

For $\gcd(n, k)=p_1^a$, $R(n, k)=((\sum p^{a-1}(C(n/p_1^{a-1}, k/p_1^{a-1})-C(n/p_1^a, k/p_1^a)))+p^aC(n/p_1^a, k/p_1^a))/k$. *For* $a=1$, $R(n, k)=(p_1^1C(n/p_1^1, k/p_1^1))/k$.

For $a=2$, $R(n, k)=(p_1^1(C(n/p_1^1, k/p_1^1)-C(n/p_1^2, k/p_1^2))+p_1^2C(n/p_1^2, k/p_1^2))/k=$ $(p_1^1C(n/p_1^1, k/p_1^1)+(p_1^2-p_1^1)C(n/p_1^2, k/p_1^2))/k$.

By induction, $X(p_1^a)=p_1^a-p_1^{a-1}$ and the only term that is not being subtracted in this chain is $p_1C(n/p_1, k/p_1)$. The same reasoning applies also to the bottom row and the vertical column in the two-dimensional matrix that has no terms underneath and to the right respectively, and also in the three-dimensional matrix to the bottom front row and right front column if the z-axis is to be constructed from the two-dimensional matrix towards the viewer. Let us now resolve for the terms whose C^R has terms that lie inside outer planes.

For $gcd(n, k)=p_1p_2$, $R(n, k)=(p_1(C(n/p_1, k/p_1)-C(n/(p_1p_2), k/(p_1p_2)))+p_2(C(n/p_2, k/p_2)-$ $C(n/(p_1p_2), k/(p_1p_2)))+p_1p_2C(n/(p_1p_2), k/(p_1p_2)))/k=(p_1C(n/p_1, k/p_1)+p_2C(n/p_2, k/p_2)+$ $(p_1p_2-p_1-p_2)C(n/(p_1p_2), k/(p_1p_2)))/k$.

For $gcd(n, k)=p_1^2p_2^2$, $R(n, k)=(p_1(C(n/p_1, k/p_1)-C(n/(p_1p_2), k/(p_1p_2))-C(n/p_1^2, k/p_1^2)+$ $C(n/(p_1^2p_2), k/(p_1^2p_2)))+p_2(C(n/p_2, k/p_2)-C(n/(p_1p_2), k/(p_1p_2))-C(n/p_2^2, k/p_2^2)+$ $C(n/(p_1p_2^2), k/(p_1p_2^2)))+p_1p_2(C(n/(p_1p_2), k/(p_1p_2))-C(n/(p_1^2p_2), k/(p_1^2p_2))-C(n/(p_1p_2^2), k/(p_1p_2^2))+$ $C(n/(p_1^2p_2^2), k/(p_1^2p_2^2)))+p_1^2(C(n/p_1^2, k/p_1^2)-C(n/(p_1^2p_2), k/(p_1^2p_2)))+p_2^2(C(n/p_2^2, k/p_2^2)-$ $C(n/(p_1p_2^2), k/(p_1p_2^2)))+p_1^2p_2(C(n/(p_1^2p_2), k/(p_1^2p_2))-C(n/(p_1^2p_2^2), k/(p_1^2p_2^2)))+$ $p_1p_2^2(C(n/(p_1p_2^2), k/(p_1p_2^2))-C(n/(p_1^2p_2^2), k/(p_1^2p_2^2)))+C(n/(p_1^2p_2^2), k/(p_1^2p_2^2)))/k=$ $(p_1C(n/p_1, k/p_1)+p_2C(n/p_2, k/p_2)+(p_1^2-p_1)C(n/p_1^2, k/p_1^2)+(p_2^2-p_2)C(n/p_2^2, k/p_2^2)+$ $(p_1p_2-p_1-p_2)C(n/(p_1p_2), k/(p_1p_2))+(p_1^2p_2-p_1^2-p_1p_2+p_1)C(n/(p_1^2p_2), k/(p_1^2p_2))+$ $(p_1p_2^2-p_2^2-p_1p_2+p_2)C(n/(p_1p_2^2), k/(p_1p_2^2))+(p_1^2p_2^2-p_1^2p_2-p_1p_2^2+p_1p_2)C(n/(p_1^2p_2^2), k/(p_1^2p_2^2)))/k$. But $p_1^2p_2-p_1^2-p_1p_2+p_1=p_1^2p_2-p_1(p_1+p_2-1)$, $p_1p_2^2-p_2^2-p_1p_2+p_2=p_1p_2^2-p_2(p_2+p_1-1)$ and $p_1^2p_2^2-p_1^2p_2-p_1p_2^2+p_1p_2=p_1^2p_2^2-p_1p_2(p_1+p_2-1)$, which is in accordance with our initial statement.

By induction, for $gcd(n, k)=p_1^ap_2^b$, where $a, b>1$, $R(n, k)=(p_1^ap_2^bC(n/(p_1^ap_2^b), k/(p_1^ap_2^b))+(\sum_a\sum_b p_1^{a-1}p_2^{b-1}(C(n/(p_1^{a-1}p_2^{b-1}), k/(p_1^{a-1}p_2^{b-1}))-$

$C(n/(p_1^a p_2^{b-1}), k/(p_1^a p_2^{b-1}))-C(n/(p_1^{a-1} p_2^b), k/(p_1^{a-1} p_2^b))+C(n/(p_1^a p_2^b), k/(p_1^a p_2^b)))+$

$(\sum_b p_1^a p_2^{b-1}(C(n/(p_1^a p_2^{b-1}), k/(p_1^a p_2^{b-1}))-C(n/(p_1^a p_2^b), k/(p_1^a p_2^b)))+$

$(\sum_a p_1^{a-1} p_2^b(C(n/(p_1^{a-1} p_2^b), k/(p_1^{a-1} p_2^b))-C(n/(p_1^a p_2^b), k/(p_1^a p_2^b)))+$

$(\sum_a p_1^{a-1}(C(n/p_1^{a-1}, k/p_1^{a-1})-C(n/(p_1^{a-1} p_2), k/(p_1^{a-1} p_2))-C(n/p_1^a, k/p_1^a)+$

$C(n/(p_1^a p_2), k/(p_1^a p_2)))+(\sum_b p_2^{b-1}(C(n/p_2^{b-1}, k/p_2^{b-1})-C(n/(p_1 p_2^{b-1}), k/(p_1 p_2^{b-1}))-$

$C(n/p_2^b, k/p_2^b)+C(n/(p_1 p_2^b), k/(p_1 p_2^b)))+p_1^a(C(n/p_1^a, k/p_1^a)-C(n/(p_1^a p_2), k/(p_1^a p_2)))+$

$p_2^b(C(n/p_2^b, k/p_2^b)-C(n/(p_1 p_2^b), k/(p_1 p_2^b))))/k$. *So,*

$X(p_1^a p_2^b)=p_1^a p_2^b+p_1^{a-1} p_2^{b-1}-p_1^a p_2^{b-1}-p_1^{a-1} p_2^b=p_1^a p_2^b-p_1^{a-1} p_2^{b-1}(p_1+p_2-1)$,

$X(p_1^a p_2)=p_1^a p_2+p_1^{a-1}-p_1^a-p_1^{a-1} p_2=p_1^a p_2-p_1^{a-1}(p_1+p_2+1)=p_1^a p_2-p_1^{a-1} p_2^0(p_1+p_2+1)$,

$X(p_1 p_2^b)=p_1 p_2^b+p_2^{b-1}-p_2^b-p_1 p_2^{b-1}=p_1 p_2^b-p_2^{b-1}(p_1+p_2+1)=p_1 p_2^b-p_1^0 p_2^{b-1}(p_1+p_2+1)$,

$X(p_1^a)=p_1^a-p_1^{a-1}$, $X(p_2^b)=p_2^b-p_2^{b-1}$, $X(p_1)=p_1$, $X(p_2)=p_2$, *confirming our initial statement and therefore covering all the terms in the two-dimensional matrix, as well as the terms on the far bottom, right and front planes of the three-dimensional matrix, if it were to be extended from the two-dimensional matrix with the z-axis pointing towards the viewer. For the remaining terms inside the three-dimensional matrix the algebraic expression will not be shown in order to save space and the following results are obtained:*

For $gcd(n, k)=p_1^1 p_2^1 p_3^1$, $R(n, k)=(p_1 C(n/p_1, k/p_1)+p_2 C(n/p_2, k/p_2)+p_3 C(n/p_3, k/p_3)+$
$(p_1 p_2-p_1-p_2)C(n/(p_1 p_2), k/(p_1 p_2))+(p_1 p_3-p_1-p_3)C(n/(p_1 p_3), k/(p_1 p_3))+$
$(p_2 p_3-p_2-p_3)C(n/(p_2 p_3), k/(p_2 p_3))+(p_1 p_2 p_3-p_1 p_3-p_2 p_3-p_1 p_2+p_1+p_2+p_3)C(n/(p_1 p_2 p_3), k/(p_1 p_2 p_3)))/k$.

For $gcd(n, k)=p_1^2 p_2^2 p_3^2$, $X(p_1)=p_1$, $X(p_2)=p_2$, $X(p_3)=p_3$, $X(p_1 p_2)=p_1 p_2-p_1-p_2$,
$X(p_1 p_3)=p_1 p_3-p_1-p_3$, $X(p_2 p_3)=p_2 p_3-p_2-p_3$, $X(p_1 p_2 p_3)=p_1 p_2 p_3+p_1+p_2+p_3-p_1 p_2-p_1 p_3-p_2 p_3$,
$X(p_1^2)=p_1^2-p_1$, $X(p_2^2)=p_2^2-p_2$, $X(p_3^2)=p_3^2-p_3$, $X(p_1^2 p_2)=p_1^2 p_2-p_1(p_1+p_2-1)$,
$X(p_1 p_2^2)=p_1 p_2^2-p_2(p_2+p_1-1)$, $X(p_1^2 p_3)=p_1^2 p_3-p_1(p_1+p_3-1)$, $X(p_1 p_3^2)=p_1 p_3^2-p_3(p_3+p_1-1)$,

$X(p_2^2p_3)=p_2^2p_3-p_2(p_2+p_3-1)$, $X(p_2p_3^2)=p_2p_3^2-p_3(p_3+p_2-1)$, $X(p_1^2p_2^2)=p_1^2p_2^2-p_1p_2(p_1+p_2-1)$,

$X(p_1^2p_3^2)=p_1^2p_3^2-p_1p_3(p_1+p_3-1)$, $X(p_2^2p_3^2)=p_2^2p_3^2-p_2p_3(p_2+p_3-1)$,

$X(p_1^2p_2p_3)=p_1^2p_2p_3-p_1(p_1p_2+p_1p_3+p_2p_3+1-p_1-p_2-p_3)$,

$X(p_1p_2^2p_3)=p_1p_2^2p_3-p_2(p_1p_2+p_1p_3+p_2p_3+1-p_1-p_2-p_3)$,

$X(p_1p_2p_3^2)=p_1p_2p_3^2-p_3(p_1p_2+p_1p_3+p_2p_3+1-p_1-p_2-p_3)$,

$X(p_1^2p_2^2p_3)=p_1^2p_2^2p_3-p_1p_2(p_1p_2+p_1p_3+p_2p_3+1-p_1-p_2-p_3)$,

$X(p_1^2p_2p_3^2)=p_1^2p_2p_3^2-p_1p_3(p_1p_2+p_1p_3+p_2p_3+1-p_1-p_2-p_3)$,

$X(p_1p_2^2p_3^2)=p_1p_2^2p_3^2-p_2p_3(p_1p_2+p_1p_3+p_2p_3+1-p_1-p_2-p_3)$, *and*

$X(p_1^2p_2^2p_3^2)=p_1^2p_2^2p_3^2-p_1p_2p_3(p_1p_2+p_1p_3+p_2p_3+1-p_1-p_2-p_3)$.

By induction, for $gcd(n, k)=p_1^ap_2^bp_3^c$, *where* $a,b,c>1$, $X(p_1)=p_1$, $X(p_2)=p_2$, $X(p_3)=p_3$,

$X(p_1^a)=p_1^a-p_1^{a-1}$, $X(p_2^b)=p_2^b-p_2^{b-1}$, $X(p_3^c)=p_3^c-p_3^{c-1}$, $X(p_1^ap_2^b)=p_1^ap_2^b-p_1^{a-1}p_2^{b-1}(p_1+p_2-1)$,

$X(p_1^ap_2)=p_1^ap_2-p_1^{a-1}p_2^0(p_1+p_2+1)$, $X(p_1p_2^b)=p_1p_2^b-p_1^0p_2^{b-1}(p_1+p_2+1)$,

$X(p_1^ap_3^c)=p_1^ap_2^c-p_1^{a-1}p_3^{c-1}(p_1+p_3-1)$, $X(p_1^ap_3)=p_1^ap_3-p_1^{a-1}p_3^0(p_1+p_3+1)$,

$X(p_1p_3^c)=p_1p_3^c-p_1^0p_3^{c-1}(p_1+p_3+1)$, $X(p_2^bp_3^c)=p_2^bp_2^c-p_2^{b-1}p_3^{c-1}(p_2+p_3-1)$,

$X(p_2^bp_3)=p_2^bp_3-p_2^{b-1}p_3^0(p_2+p_3+1)$, $X(p_2p_3^c)=p_2p_3^c-p_2^0p_3^{c-1}(p_2+p_3+1)$,

$X(p_1^ap_2p_3)=p_1^ap_2p_3-p_1(p_1p_2+p_1p_3+p_2p_3+1-p_1-p_2-p_3)$,

$X(p_1p_2^bp_3)=p_1p_2^bp_3-p_2(p_1p_2+p_1p_3+p_2p_3+1-p_1-p_2-p_3)$,

$X(p_1p_2p_3^c)=p_1p_2p_3^c-p_3(p_1p_2+p_1p_3+p_2p_3+1-p_1-p_2-p_3)$,

$X(p_1^ap_2^bp_3)=p_1^ap_2^bp_3-p_1p_2(p_1p_2+p_1p_3+p_2p_3+1-p_1-p_2-p_3)$,

$X(p_1^ap_2p_3^c)=p_1^ap_2p_3^c-p_1p_3(p_1p_2+p_1p_3+p_2p_3+1-p_1-p_2-p_3)$,

$X(p_1p_2^bp_3^c)=p_1p_2^bp_3^c-p_2p_3(p_1p_2+p_1p_3+p_2p_3+1-p_1-p_2-p_3)$, *and*

$X(p_1^ap_2^bp_3^c)=p_1^ap_2^bp_3^c-p_1p_2p_3(p_1p_2+p_1p_3+p_2p_3+1-p_1-p_2-p_3)$, *which is all in accordance with our initial statement.*

Q.E.D.

Corollary 3.2 Following the proof in Proposition 2, it can be postulated that $X(d)$ will remain unchanged for any $\omega(\gcd(n, k))>3$, where ω counts the number of unique prime factors of a number.

Proposition 2.2 For the necklace formula, where $\omega(\gcd(n, k))<4$,

$N(n, k)=1/k \sum_{d|\gcd(n, k)} Y(d)C(n/d, k/d)$, where

$Y(d)=1 \Leftrightarrow d=1$

$Y(d)=X(d)-1 \Leftrightarrow d=p^1$

$Y(d)=X(d) \Leftrightarrow d=p^a$ and $a>1$

$Y(d)=X(d)+1 \Leftrightarrow d=p_1^1p_2^1$

$Y(d)=X(d) \Leftrightarrow d=p_1^ap_2^b$ and a or $b>1$

$Y(d)=X(d)-1 \Leftrightarrow d=p_1^1p_2^1p_3^1$

$Y(d)=X(d) \Leftrightarrow d=p_1^ap_2^bp_3^c$ and either a,b or $c>1$.

Proof. *We established in Corollary 1.2 that* $N(n, k)=L(n, k)+R(n, k)$. *But we also know that* $L(n, k)=(C(n, k)-C^R(n, k))/k$ *and from Proposition 2.1 we obtained the formula* $R(n, k)=1/k \sum_{d|\gcd(n, k), d>1} X(d)C(n/d, k/d)$. *By combining these formulas, we obtain* $N(n, k)=1/k(C(n, k)-C^R(n, k)+\sum_{d|\gcd(n, k), d>1} X(d)C(n/d, k/d))$. *We can now proceed with solving each case.*

For $\gcd(n, k)=p_1$, $C^R(n, k)=C(n/p_1, k/p_1)$, *so* $N(n, k)=1/k(C(n, k)-C(n/p_1, k/p_1)+X(p_1)C(n/p_1, k/p_1))=1/k(C(n, k)+(X(p_1)-1)C(n/p_1, k/p_1))$.

For $\gcd(n, k)=p_1^a$, $C^R(n, k)=C(n/p_1, k/p_1)$, *so* $N(n, k)=1/k(C(n, k)-C(n/p_1, k/p_1)+X(p_1)C(n/p_1, k/p_1)+\sum_{a>1} X(p_1^a)C(n/p_1^a, k/p_1^a))=1/k(C(n, k)+(X(p_1)-1)C(n/p_1, k/p_1)+\sum_{a>1} X(p_1^a)C(n/p_1^a, k/p_1^a))$.

For $gcd(n, k)=p_1p_2$, $C^R(n, k)=C(n/p_1, k/p_1)+C(n/p_2, k/p_2)-C(n/(p_1p_2), k/(p_1p_2))$, *therefore* $N(n, k)=1/k(C(n, k)-C(n/p_1, k/p_1)-C(n/p_2, k/p_2)+C(n/(p_1p_2), k/(p_1p_2))+X(p_1)C(n/p_1, k/p_1)+X(p_2)C(n/p_2, k/p_2)+X(p_1p_2)C(n/(p_1p_2), k/(p_1p_2)))=1/k(C(n, k)+(X(p_1)-1)C(n/p_1, k/p_1)+(X(p_2)-1)C(n/p_2, k/p_2)+(X(p_1p_2)+1)C(n/(p_1p_2), k/(p_1p_2)))$.

For $gcd(n, k)=p_1^ap_2^b$, $C^R(n, k)=C(n/p_1, k/p_1)+C(n/p_2, k/p_2)-C(n/(p_1p_2), k/(p_1p_2))$, *therefore* $N(n, k)=1/k(C(n, k)-C(n/p_1, k/p_1)-C(n/p_2, k/p_2)+C(n/(p_1p_2), k/(p_1p_2))+X(p_1)C(n/p_1, k/p_1)+X(p_2)C(n/p_2, k/p_2)+(\sum_{a>1} X(p_1^a)C(n/p_1^a, k/p_1^a))+(\sum_{b>1} X(p_2^b)C(n/p_2^b, k/p_2^b))+X(p_1p_2)C(n/(p_1p_2), k/(p_1p_2))+(\sum_{a>1}\sum_{b>1} X(p_1^ap_2^b)C(n/(p_1^ap_2^b), k/(p_1^ap_2^b)))+(\sum_{a>1} X(p_1^ap_2)C(n/(p_1^ap_2), k/(p_1^ap_2)))+(\sum_{b>1} X(p_1p_2^b)C(n/(p_1p_2^b), k/(p_1p_2^b))))=1/k(C(n, k)+(X(p_1)-1)C(n/p_1, k/p_1)+(X(p_2)-1)C(n/p_2, k/p_2)+(X(p_1p_2)+1)C(n/(p_1p_2), k/(p_1p_2))+(\sum_{a>1} X(p_1^a)C(n/p_1^a, k/p_1^a))+(\sum_{b>1} X(p_2^b)C(n/p_2^b, k/p_2^b))+(\sum_{a>1}\sum_{b>1} X(p_1^ap_2^b)C(n/(p_1^ap_2^b), k/(p_1^ap_2^b)))+(\sum_{a>1} X(p_1^ap_2)C(n/(p_1^ap_2), k/(p_1^ap_2)))+(\sum_{b>1} X(p_1p_2^b)C(n/(p_1p_2^b), k/(p_1p_2^b))))$ *and* $X(p_1^ap_2^b)=X(p_1^ap_2)=X(p_1p_2^b)$.

For $gcd(n, k)=p_1p_2p_3$, $C^R(n, k)=C(n/p_1, k/p_1)+C(n/p_2, k/p_2)+C(n/p_3, k/p_3)+C(n/(p_1p_2p_3), k/(p_1p_2p_3))-C(n/(p_1p_2), k/(p_1p_2))-C(n/(p_1p_3), k/(p_1p_3))-C(n/(p_2p_3), k/(p_2p_3))$, *therefore* $N(n, k)=1/k(C(n, k)+(X(p_1)-1)C(n/p_1, k/p_1)+(X(p_2)-1)C(n/p_2, k/p_2)+(X(p_3)-1)C(n/p_3, k/p_3)+(X(p_1p_2)+1)C(n/(p_1p_2), k/(p_1p_2))+(X(p_1p_3)+1)C(n/(p_1p_3), k/(p_1p_3))+(X(p_2p_3)+1)C(n/(p_2p_3), k/(p_2p_3))+(X(p_1p_2p_3)-1)C(n/(p_1p_2p_3), k/(p_1p_2p_3)))$.

For $gcd(n, k)=p_1^ap_2^bp_3^c$, $X(p_1^ap_2^b)=X(p_1^ap_2)=X(p_1p_2^b)$, $X(p_1^ap_3^c)=X(p_1^ap_3)=X(p_1p_3^c)$, $X(p_2^bp_3^c)=X(p_2^bp_3)=X(p_2p_3^c)$, *and* $X(p_1^ap_2^bp_3^c)=X(p_1^ap_2^bp_3)=X(p_1^ap_2p_3^c)=X(p_1p_2^bp_3^c)=X(p_1p_2p_3^c)=X(p_1p_2^bp_3)=X(p_1^ap_2p_3)$, *thereby confirming the initial statement.*

Q.E.D.

Corollary 3.3 As we have shown in Corollary 3.1, $(n!/(k!(n-k)!))/n=$

$((n\text{-}1)!/((k\text{-}1)!(n\text{-}k)!))/k$ and we are taking the same divisors into account as the old formula, therefore, $\varphi(d) \equiv Y(d)$ *and* for $\omega(\gcd(n, k))>3$, we can use Euler's totient function $\varphi(d)$ instead of $Y(d)$: $N(n, k)=1/k \sum_{d|\gcd(n, k)} \varphi(d)C(n/d, k/d)$, which has already been proven by mathematicians.

Corollary 3.4 Following Corollary 3.2, it can be postulated that $Y(d)$ will remain unchanged for any $\gcd(n, k)$ and $X(d) \equiv \varphi(d)\text{-}\mu(d)$.

We have now finished proving the formulas needed to calculate necklaces. By applying the formulas for our 12 tone system in music, we obtain the following numbers of chord types:

Table 3. Results for calculating the number of chord types

Number of notes in a chord (k)	Number of chord types
2 notes	6
3 notes	19
4 notes	43
5 notes	66
6 notes	80
7 notes	66
8 notes	43
9 notes	19

10 notes	6
11 notes	1
12 notes	1

As opposed to chord type inversions (circular combinations), we notice that $N(n, k)=N(n, n-k)$. In the following table we recapitulate the musical terms (with the equivalent mathematical terms in brackets) and their total numbers for $2 \leq k \leq 12$:

Table 4. Total number of pitch-class sets

Chords (linear combinations)	variable
Chord type permutations (circular permutations)	108,505,111
Chord type inversions (circular combinations)	2,047
Chord types (binary necklaces)	350

Let us now use the formula for rotationally symmetric necklaces to calculate the number of symmetric chord types, also known as modes of limited transposition:

Table 5. Number of rotationally symmetric chord types

Number of	Number of	Conventional names for the chord types

notes in a chord (*k*)	symmetric chord types	
2 notes	1	tritone
3 notes	1	augmented triad
4 notes	3	diminished seventh chord, dom7 b5, maj7 sus4 b5
5 notes	0	
6 notes	5	augmented scale, whole-tone scale, fifth mode, dom7 b5 #9 13, tritone scale
7 notes	0	
8 notes	3	whole-half scale, fourth mode, sixth mode
9 notes	1	third mode
10 notes	1	seventh mode
11 notes	0	
12 notes	1	chromatic scale

2. Chord progressions

We will now bring our attention to the theme of combinations and progressions of chords.

Definition 3.1 A chord progression is the movement of one chord to another.

As musicians we would like to know in how many ways we can combine chords to form chord progressions without repeating ourselves through transposition. Before that, we need to consider some further properties of these chords that will prove useful in the following chapter.

2.1 Degree of rotational symmetry

We have already established the difference between symmetrical and asymmetrical necklaces denoted with R and L respectively. In the case of asymmetrical chords, we know that there are 12 different transpositions of a given chord, whereas rotationally symmetric chords have less than 12, depending on the common divisors of *n* and *k*. So the number of times a necklace can be rotated until it produces its identity determines the degree of symmetry or how symmetrical a chord can be. For example, the augmented triad can be transposed 4 times until it produces the same chord given inversional equivalence. The diminished seventh chord on the other hand can be transposed 3 times until it produces the same chord.

Let R(*n, k*) be the set of rotationally symmetric necklaces of length *n* and *k* objects. Let R^d be the subset of chords from that given set with a *d* degree of symmetry, such that any necklace from that subset can be rotated *n*/*d* times before it produces its identity (*d* corresponds to the common divisor by which that particular necklace can be divided). For example, the augmented triad is part of R^3 from R(12, 3) because gcd(12, 3)=3 and it can be rotated 12/3=4 times until it produces the same chord. Let $T(N_x^d)$ be the number of transpositions for a given chord $N_x^d \in N$, such that $N = L \cup R$ (the set of all chords). Then $T(L_x)=12$ and $T(R_x^d)=12/d$.

Now we can enumerate the rotationally symmetric chords in terms of their degree of symmetry:

Table 6. Rotationally symmetric chord types

Degree of symmetry	Chord types
R^2	tritone, dom7 b5, maj7 sus4 b5, fifth mode, dom7 b5 #9 13, tritone scale, fourth mode, sixth mode, seventh mode
R^3	augmented triad, augmented scale, third mode
R^4	diminished seventh, whole-half scale
R^6	whole-tone scale
R^{12}	chromatic scale

2.2 Number of chord type progressions

Proposition 3.1 Let N be the total number of chord types (necklaces) corresponding to the cardinality of N. Let P_j be the number of chord type progressions of length j (having j chords). Then $P_j = N(\sum T(\mathrm{N}_x^d))^\wedge(j\text{-}1)$.

Proof. *We begin a chord progression by choosing any chord from the* N *set. The particular transposition of that chord is irrelevant at the beginning since we are interested in the intervals between chords. By choosing C maj → G maj or D maj → A maj, we end up with the same type of chord progression, which can be interpreted as a I-V progression in a given context. Therefore, when selecting the first chord in the chain, it suffices to select the chord type transposed arbitrarily. Suppose we start the progression with a C maj. For the* *second chord, it does make a difference if we choose a D maj or a B maj, given that the maj chord is asymmetric and each of the 12 transpositions will produce a different combination with the first chord. However, in the case of rotationally symmetric chords, we have already established that it takes less than 12 transpositions to produce the same chord. For example, E aug ≡ G# aug ⇒ C maj → E aug ≡ C maj → G# aug. So for each chord type in the beginning of the progression, there are all the available chord types multiplied with their transpositions depending on the degree of symmetry. This results in* $P_2 = N\sum T(N_x^d)$*. For the third chord, fourth chord etc. in the progression the number of options stays the same, i.e.* $\sum T(N_x^d)$*, since choosing the same chord again as previously is accepted according to Definition 3.1. So for a progression with j number of chords,*

$P_j = N(\sum T(N_x^d))(\sum T(N_x^d))\ldots(\sum T(N_x^d))$ *j-1 number of times, which results in*

$P_j = N(\sum T(N_x^d))\text{^}(j-1)$.

Q.E.D.

To give an idea about how large P_j is, we can compute the formula for j=4:

P_4=350(12*334+6*9+4*3+3*2+2+1)^3=23,823,533,925,450.

A potentially more useful question is how many chord type progressions there are, such that no two identical chords are played consecutively.

Corollary 3.5 Following the proof in Proposition 3.1, after choosing the first chord in the progression, there are $(\sum T(N_x^d))-1$ possibilities for the second chord, because you cannot play the same chord as before. This leads to $P_j = N((\sum T(N_x^d))-1)\text{^}(j-1)$.

Computing P_4 after taking this restriction into account results in:

P_4=350(12*334+6*9+4*3+3*2+2)^3=23,806,033,778,800.

It would take roughly 1.3 billion chord type progression every day for 50 years to search all of them. But suppose that you know which chord type the progression should start with because listening to all 350 chord types is not very difficult and makes for an easier pick. Then P_4=1(12*334+6*9+4*3+3*2+2)^3=68,017,239,368. This reduces the search to roughly 3.5 million chord type progressions every day for 50 years, which is still unrealistic. However, on a practical level, musicians do not need all 350 necklaces for their music. They might not prefer chromatic clusters or any chord with more than 6 notes and this restriction reduces the number of possible chord type progressions drastically. For example, suppose we only choose the following set of triads: N={maj, min, dim, aug, sus}. Then the number of chord type progressions with 4 chords is computed as follows: P_4=5(12*4+4*1)^3=703,040. However, if you already know which chord the progression should start with, then P_4=1(12*4+4*1)^3=140,608. An even faster way of navigating all of the possibilities is to just study pairs of chords, remember their characteristic sound and then chain them together. In our example with N=5, you would just need to study P_2=5(12*4+4*1)=260. This is a realistically small list of pairs that one can remember over the years. Consider the set N consisting of all 19 triads. Then P_2=19(12*18+4*1)=4180. This is a very powerful and efficient way of navigating through chord progressions, given the huge variation of combinations. To make this navigation easier, I made a website that allows you to select and combine chord types to form progressions and export them as MIDI or audio files: chordprogressions.org

Disclosure statement

The author reports there are no competing interests to declare.

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